SINGULAR PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS WITH TIME DEPENDENT COEFFICIENTS

BY

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Introduction. In this paper we consider the general operator

(0.1) $$\Omega(t) = D_{\mu(t)}D_{\sigma(t)} + c(x,t) \qquad \begin{array}{l} a \leqq t \leqq b, \\ r_1 \leqq x \leqq r_2. \end{array}$$

In (0.1) for each t, $\mu(t) = \mu_t$ is a Borel measure defined on the Borel sets of (r_1,r_2), $\sigma(t) = \sigma_t$ is a strictly increasing function continuous on (r_1,r_2), $c(x,t) \leqq 0$, and $c(x,t)$ is continuous in x on $[r_1,r_2]$ for each $t \in [a,b]$. The generalized derivatives $D_{\sigma(t)}$ and $D_{\mu(t)}D_{\sigma(t)}$ are defined in §2. The operator $\Omega(t)$ is, for each $t \in [a,b]$, a linear, but not necessarily bounded, operator. The purpose of this paper is to present conditions on μ_t and σ_t such that given an initial value for $t = a$ and possible boundary conditions at r_1 and r_2 there exists a unique solution of the equation

(0.2) $$u_t(x,t) = D_{\mu(t)}D_{\sigma(t)}u(x,t) + c(x,t)u(x,t) \qquad \begin{array}{l} a \leqq t \leqq b, \\ r_1 \leqq x \leqq r_2. \end{array}$$

Four types of boundaries, regular, entrance, exit, and natural are considered. These types are defined in §2. The boundary classifications of (0.1) for the σ_t and μ_t independent of t were formulated by K. Itô and used by H. McKean [11] in accordance with W. Feller [4], and the boundary conditions (2.8) are the generalized classical ones.

W. Feller [3; 2] has shown that for each t, $\Omega(t)$ is a generalization of the classical differential operator

(0.3) $$\Omega(t) = a(x,t)\frac{d^2}{dx^2} + b(x,t)\frac{d}{dx} + c(x,t) \qquad \begin{array}{l} a \leqq t \leqq b, \\ r_1 \leqq x \leqq r_2. \end{array}$$

(See §2 for the pertinent details of this demonstration.) In (0.3) for each t, $a(x,t) > 0$, $a(x,t)$ and $b(x,t)$ are continuous in x on (r_1,r_2), $c(x,t) \leqq 0$, and $c(x,t)$ is continuous in x on $[r_1,r_2]$. The operators we consider are allowed to be singular in that $a(x,t)$ and $b(x,t)$ are allowed to have discontinuities in x at the boundaries r_1 and r_2 and $a(x,t)$ may vanish at the boundaries. The boundary classifications are seen to depend on the behavior of $b^*(x,t) = b(x,t)/a(x,t)$ and the boundary con-

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ditions are the classical ones. Our work exhibits conditions on the coefficients of the equation

$$(0.4) \qquad\qquad u_t(x,t) = \Omega(t)\,u\,(x,t)$$

which will guarantee the existence of a solution satisfying the classical boundary conditions and uniquely determined by an initial value $u(x,a)$.

Some advantages of the generalized differential operator (0.1) are discussed by W. Feller [2] for the time independent case. The generalization has applications in probability theory and may be used to advantage even for completely classical problems. As W. Feller has pointed out, the generalization achieves a considerable simplification and unification of the theory. In applications one frequently finds that for each t the coefficients of equation (0.4) have jump discontinuities. In such cases one has to piece together solutions in various intervals. But W. Feller [2] has noted that the simple expedient of writing the operator $\Omega(t)$ in the form (0.1) reduces the whole to one single equation and eliminates the laborious piecemeal work. A specific example of such a problem appears as Example 4 in §8.

Equation (0.2) can be considered as a special case of the so-called equation of evolution

$$(0.5) \qquad\qquad \frac{du(t)}{dt} = A(t)u(t) \qquad\qquad a \le t \le b$$

where for each $t \in [a,b]$, $u(t)$ is an element of a Banach space X and $A(t)$ is a linear operator. We shall take as underlying Banach space X either $C[r_1,r_2]$ or a subspace of $C[r_1,r_2]$ depending on the boundary conditions (Definition (2.1)). The domain of our operator $\Omega(t)$ also depends on the boundary conditions. It is defined as the set of all $f \in X$ which satisfy appropriate boundary conditions and which are such that $\Omega(t)f$ is itself in X.

T. Kato [9] has constructed the unique solution to the equation of evolution subject to rather general hypotheses on $A(t)$. In this paper we determine conditions on the scale σ_t and measure μ_t which guarantee that $\Omega(t)$ satisfies Kato's hypotheses. We supplant Kato's set of hypotheses by a set of conditions which are slightly stronger than Kato's although somewhat more convenient for our purposes. T. Kato [10] has anounced the solution of (0.5) under less stringent conditions than those in [9].

Considerable work has been done by W. Feller [4], J. Elliott [1], and E. Hille [7] in constructing elementary solutions for the classical operator (0.3) where $\Omega(t) = \Omega$ is independent of t and by H. McKean [11] for the generalized operator (0.1) in the t independent case. The elementary solution for $\Omega(t)$ with time dependent coefficients subject to a variety of conditions has been constructed by W. Feller [5]. It appears that our conditions on $b(x,t)$ are less stringent than Feller's as far as x is concerned but are somewhat more restrictive in t.

The method of proof which we use excludes the possibility of using boundary conditions which are dependent on t because in Kato's hypotheses $\mathrm{Dmn}\, A(t)$ is independent of t and our $\mathrm{Dmn}\, \Omega(t)$ depends on the boundary conditions. Examples of particular operators $\Omega(t)$ with coefficients $b(x,t)$ having discontinuities at boundaries but which still satisfy the conditions of our main theorem are given in §8.

In §1 we state Kato's theorem and give our conditions which supplant Kato's hypotheses. §2 is devoted to preliminary groundwork and the statement of important definitions and conditions. The development of our main theorem for the reduced operator appears in §§3–5, and in §6 we indicate how these proofs go through for the more general case in which $c(x,t)$ is negative. In §7 we show in detail how our main theorem applies to the special case of the classical operator, and §8 consists of several specific examples.

I wish to express my gratitude to Joanne Elliott for suggesting this topic and for her most helpful guidance.

1. **Kato's theorem.** We wish to employ some of the main results in a paper of T. Kato's [9]. Before summarizing these results, it is convenient to make some preliminary definitions.

DEFINITION 1.1. An operator A in a Banach space X is said to have property S on X: (i) if A is a closed linear operator with domain dense in X; (ii) if the resolvent set of A includes all positive reals; and (iii) if $\| (I - \lambda A)^{-1} \| \leqq 1$ for $\lambda > 0$. Let the following conditions be given:

K$_1$. $A(t)$ is defined for $t \in [a,b]$ and has property S for each t.

K$_2$. The domain of $A(t)$ is independent of t.

K$_3$. $C(t,s) = [I - A(t)][I - A(s)]^{-1}$ is uniformly bounded; that is, there is an $M > 0$ such that $\| C(t,s) \| \leqq M$ for every s,t.

K$_4$. $C(t,s)$ is of bounded variation in t in the sense that there is an $N \geqq 0$ such that

$$\sum_{j=1}^{n} \| C(t_j,s) - C(t_{j-1},s) \| \leqq N$$

for every partition $a = t_0 < t_1 < \cdots < t_n = b$ of the interval $[a,b]$, for some s.

K$_5$. $C(t,s)$ is weakly continuous in t for some s.

K$_6$. For some s, $C(t,s)$ is weakly differentiable in t and the weak derivative $\partial C(t,s)/\partial t$ is strongly continuous in t.

The following theorem represents part of Kato's results:

THEOREM 1.1. *If conditions* K$_1$, K$_2$, $\cdots$, K$_6$ *are satisfied, then there exists a solution of the equation*

$$dx(t)/dt = A(t)x(t) \qquad\qquad a \leqq t \leqq b,$$

where for each $t \in [a,b]$, $x(t)$ *is an element of a complex Banach space* X *and* $A(t)$ *is a linear operator in* X. *The solution is uniquely determined by the initial value* $x(a)$ *and is strongly continuously differentiable. Furthermore, if the solution is formally given as* $x(t) = U(t,a)x(a)$, *then* $U(t,t) = I$ *and* $U(t, r) = U(t,s) U(s,r)$, $r \leqq s \leqq t$.

For our purposes it is convenient to supplant the hypotheses of Kato's theorem with the following set of somewhat stronger conditions:

C_1. $A(t)$ is defined for $t \in [a,b]$ and has property S for each t.

C_2.   The domain of $A(t)$ is independent of t.

C_3. $C(t,s) = [I - A(t)][I - A(s)]^{-1}$ is strongly continuously differentiable in t for some s. That is, for some s the strong derivative $C_t(t,s)$ of $C(t,s)$ exists and is strongly continuous in t.

This is accomplished by proving the following theorem:

THEOREM 1.2.   *Conditions* C_1, C_2 *and* C_3 *imply Kato's conditions* $K_1, K_2, \cdots, K_6$.

Proof. It suffices to prove that conditions C_1, C_2 and C_3 imply Kato's conditions K_3 and K_4. The uniform boundedness of $C(t,s)$ follows readily (cf. T. Kato [9, pp. 215-216]). Futhermore, $C(t,s)$ is of bounded variation in t since for some fixed $s = s_0$, $\| C_t(t,s_0) \|$ is uniformly bounded in t.

2. **Preliminary definitions.** Let x be an element of the open interval (r_1,r_2) and for convenience let $0 \in (r_1,r_2)$. Let $t \in [a,b]$ for $a \geqq 0$. For each t let $\sigma_t(x) = \sigma(x,t)$ be a strictly increasing continuous function—that is, an arbitrary scale. For each t let μ_t be a Borel measure defined on the class of Borel sets of (r_1,r_2) and let $\mu(x,t) = \mu_t((0,x])$ for $x > 0$, $\mu(x,t) = \mu_t((x,0])$ for $x < 0$ and $\mu(0,t) = \mu_t(\{0\})$. $\mu(x,t)$ is right continuous in x for each t. $\mu(x,t)$ is monotone, and we assume $\mu(x,t)$ to be strictly increasing for $x > 0$ and strictly decreasing for $x < 0$ so that its discontinuities form a Borel set of measure zero.

Let $f \in C[r_1,r_2]$. For each t and $x \in (r_1,r_2)$ define

$$(2.1) \qquad D_{\sigma(t)}f(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{\sigma_t(x + h) - \sigma_t(x)}$$

and at points of continuity of $\mu(x,t)$ define

$$D^+_{\mu(t)}D_{\sigma(t)}f(x) = \lim_{h \to 0^+} \frac{D_{\sigma(t)}f(x + h) - D_{\sigma(t)}f(x)}{\mu_t((x,x + h))}$$

and

$$D^-_{\mu(t)}D_{\sigma(t)}f(x) = \lim_{h \to 0^-} \frac{D_{\sigma(t)}f(x + h) - D_{\sigma(t)}f(x)}{-\mu_t((x + h,x))}.$$

If $D^+_{\mu(t)}D_{\sigma(t)}f(x) = D^-_{\mu(t)}D_{\sigma(t)}f(x)$, then define

$$(2.2) \qquad D_{\mu(t)}D_{\sigma(t)}f(x) = D^+_{\mu(t)}D_{\sigma(t)}f(x) = D^-_{\mu(t)}D_{\sigma(t)}f(x).$$

At points of discontinuity of $\mu(x,t)$ define

$$(2.3) \qquad D_{\mu(t)}D_{\sigma(t)}f(x) = \frac{D_{\sigma(t)}f(x+) - D_{\sigma(t)}f(x-)}{\mu(x+,t) - \mu(x-,t)}.$$

Of course $D_{\sigma(t)}f$ is continuous with respect to μ_t when $D_{\mu(t)}D_{\sigma(t)}f$ exists. The only possible points of discontinuity of $D_{\sigma(t)}f$ are the points of discontinuity of μ_t. For convenience we will assume that $D_{\sigma(t)}f(x)$ is continuous to the right for $x \in [r_1, r_2]$.

Let $\Omega(t)f(x) = D_{\mu(t)}D_{\sigma(t)}f(x) + c(x,t)f(x)$, where for each t, $c(x,t) \leqq 0$ and is continuous in x on the closed interval $[r_1, r_2]$. The domain of $\Omega(t)$ will be defined below. We wish to consider the equation

$$(2.4) \qquad u_t(x,t) = \Omega(t)u(x,t).$$

To that end let us classify boundaries and define the classical boundary conditions.

As a preliminary to classifying the boundaries, let us imitate H. P. McKean [11] in defining[1]

$$(2.5) \qquad J_i(t) = (-1)^i \int_0^{r_i} (1 - c(x,t))\mu(x,t)d\sigma_t$$

and

$$(2.6) \qquad K_i(t) = (-1)^i \int_0^{r_i} (1 - c(x,t))\,|\sigma(x,t)|\,d\mu_t \qquad \text{for } i = 1,2.$$

For future use note that for each t

$$(2.7) \qquad \begin{aligned} K_i(t) < +\infty &\to K_i^0(t) = (-1)^i \int_0^{r_i} |\sigma(x,t)|\,d\mu_t \\ &< +\infty \to (-1)^i \int_0^{r_i} d\mu_t < +\infty \end{aligned}$$

and

$$(2.8) \qquad \begin{aligned} J_i(t) < +\infty &\to J_i^0(t) = (-1)^i \int_0^{r_i} \mu(x,t)d\sigma_t \\ &< +\infty \to (-1)^i \int_0^{r_i} d\sigma_t < +\infty. \end{aligned}$$

Here the $K_i(t)$, $K_i^0(t)$, $J_i(t)$ and $J_i^0(t)$ are all non-negative for $i = 1,2$; and of course if $c(x,t) \equiv 0$, $K_i(t) = K_i^0(t)$ and $J_i(t) = J_i^0(t)$. As suggested by W. Feller [4] and H. P. McKean and K. Itô [11], we now *classify boundaries* as follows: where $i = 1$ or 2,

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[1] Throughout this paper the notation $\int^{x_2} f(x,t)d\mu_t$ is used for the integral over the semi-open interval $(x_1, x_2]$.

$$[r_i \text{ is regular}] \leftrightarrow [\text{for each } t, J_i(t) < +\infty \text{ and } K_i(t) < +\infty],$$

(2.9)
$$[r_i \text{ is exit}] \quad \leftrightarrow [\text{for each } t, J_i(t) < +\infty \text{ and } K_i(t) = +\infty],$$

$$[r_i \text{ is entrance}] \leftrightarrow [\text{for each } t, J_i(t) = +\infty \text{ and } K_i(t) < +\infty],$$

$$[r_i \text{ is natural}] \leftrightarrow [\text{for each } t, J_i(t) = +\infty \text{ and } K_i(t) = +\infty].$$

Define the *classical boundary conditions* as follows: where $i = 1$ or 2 and $p_i \in [0, 1]$,

$$\lim_{x \to r_i} \{(1 - p_i)u(x, t) + (-1)^i p_i D_{\sigma(t)} u(x, t)\} = 0 \text{ for } r_i \text{ regular},$$

(2.10)
$$\lim_{x \to r_i} u(x, t) = 0 \text{ for } r_i \text{ exit},$$

$$\lim_{x \to r_i} D_{\sigma(t)} u(x, t) = 0 \text{ for } r_i \text{ entrance},$$

$$\lim_{x \to r_i} u(x, t) = 0 \text{ for } r_i \text{ natural}.$$

DEFINITION 2.1. For each t we associate with the operator $\Omega(t)$ a Banach space X given by: (i) $C[r_1, r_2]$ if r_i ($i = 1, 2$) is entrance or regular with $p_i \neq 0$ in (2.10); (ii) the subspace of $C[r_1, r_2]$ of functions vanishing at r_i if r_i is regular with $p_i = 0$ in (2.10) or if r_i is exit or natural.

We may now precisely define the operator $\Omega(t)$ as follows: if

(2.11)
$$\Omega(t) = D_{\mu(t)} D_{\sigma(t)} + c(x, t),$$

then *Domain of $\Omega(t)$ is the set of all $u \in X$ such that $u(x)$ satisfies the boundary conditions prescribed for the type of boundaries in question and such that*[2] $\Omega(t) u \in X$.

Special Case. For each t the operator

(2.12)
$$A(t) = a(x, t) \frac{d^2}{dx^2} + b(x, t) \frac{d}{dx} + c(x, t),$$

is a special case of (2.11) when $a(x, t) > 0$, $a(x, t)$ and $b(x, t)$ are continuous for $x \in (r_1, r_2)$, and $c(x, t) \leqq 0$ and continuous on $[r_1, r_2]$. This was shown by W. Feller in his paper *On second order differential operators* [3]. (See also W. Feller [2].) In particular consider the so-called reduced operator

$$A(t) = a(x, t) \frac{d^2}{dx^2} + b(x, t) \frac{d}{dx}.$$

If $b(x, t)/a(x, t) = b^*(x, t)$, and if

(2.13)
$$B(x, t) = \int_0^x b^*(s, t) ds,$$

then by choosing

(2.14)
$$\sigma(x, t) = \int_0^x e^{-B(s, t)} ds$$

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(2) $\Omega(t)u \in X$ by virtue of the fact that $\Omega(t)u \in C(r_1, r_2)$, $\lim_{x \to r_i} \Omega(t)u(x) = L_i$ exists finite, and $\Omega(t)u(r_i)$ is defined as L_i.

and

$$(2.15) \qquad \mu(x,t) = \pm \int_0^x \frac{1}{a(s,t)} e^{B(s,t)} \, ds,$$

according as $x > 0$ or $x < 0$, the reduced operator is readily seen to be of the form $D_{\mu(t)}D_{\sigma(t)}$. Examples of the reduced operator with specific coefficients will be presented in §8.

The following lemma is of fundamental importance. It was proved for the operator

$$A(t) = a(x,t) \frac{d^2}{dx^2} + b(x,t) \frac{d}{dx} + c(x,t)$$

by W. Feller [4]; H. P. McKean [11] used direct analogy with Feller's proof to establish the lemma for the operator (2.11).

LEMMA 2.1. *For each $t \in [a,b]$ let $\sigma_t(x) = \sigma(x,t)$ be a strictly increasing continuous function. For each $t \in [a,b]$ let μ_t be a Borel measure defined on the Borel sets of (r_1, r_2), and let $\mu(x,t) = \mu_t((0,x])$ for $x > 0$ and $\mu(x,t) = \mu_t((x,0])$ for $x < 0$. Assume $\mu(x,t)$ is strictly increasing for $x > 0$ and strictly decreasing for $x < 0$. Let $c(x,t) \leqq 0$ be continuous in x on $[r_1, r_2]$ for each $t \in [a,b]$. Then in all boundary classifications condition C_1 holds for the operator (2.11), that is, (2.11) is defined for $t \in [a,b]$ and has property S on X for each t.*

Throughout this entire paper we shall assume that the following condition is true:

A. *For each $t \in [a,b]$, μ_t and σ_t are differentiable with respect to μ_s and σ_s respectively. That is, the derivatives*

$$\frac{d\sigma_t}{d\sigma_s} = \lim_{h \to 0} \frac{\sigma_t(x+h) - \sigma_t(x)}{\sigma_s(x+h) - \sigma_s(x)},$$

$$\frac{d\mu_t}{d\mu_s} = \lim_{h \to 0} \frac{\mu_t((x,x+h))}{\mu_s((x,x+h))} \quad \text{at points of continuity of } \mu(x,s),$$

and

$$\frac{d\mu_t}{d\mu_s} = \frac{\mu(x^+,t) - \mu(x^-,t)}{\mu(x^+,s) - \mu(x^-,s)} \quad \text{at points of discontinuity of } \mu(x,s) \text{ exist.}$$

In addition to condition A stated above, future reference will be made to several other conditions on μ_t and σ_t. These conditions are stated in terms of[3]

$$(2.16) \qquad \phi(x,s,t) = \frac{d\mu_t}{d\mu_s} \cdot D\mu_{(t)}\left(\frac{d\sigma_t}{d\sigma_s}\right) = D\mu_{(s)}D\sigma_{(s)}\sigma(t,x),$$

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[3] If for $t,s \in [a,b]$, $f(x,t,s)$ is continous in x on $[r_1, r_2]$, then for each t,s, $f(x,t,s)$ defines a vector-valued function from $a \leqq s \leqq b$, $a \leqq t \leqq b$ to $C[r_1, r_2]$; and we denote the value of the function by $f(\cdot,t,s)$ so $f(\cdot,t,s)$ is an element of $C[r_1, r_2]$. In particular, (2.16) and (2.17) define the real-valued functions $\phi(\cdot,s,t)$ and $\psi(\cdot,s,t)$.

(2.17)
$$\psi(x,s,t) = \frac{d\mu_t}{d\mu_s} \cdot \frac{d\sigma_t}{d\sigma_s},$$

(2.18)
$$C(t,s) = [I - \Omega(t)][I - \Omega(s)]^{-1},$$

(2.19)
$$F(s,x) = [I - \Omega(s)]^{-1} f(x).$$

For the convenience of the reader we shall conclude this section with a list of the other conditions on μ_t and σ_t to which we shall refer in the sequel.

   B.   For each $s \in [a,b]$, μ_s is differentiable with respect to σ_s, that is,

$$\frac{d\mu_s}{d\sigma_s} = \lim_{h \to 0} \frac{\mu_s(x, x+h)}{\sigma_s(x+h) - \sigma_s(x)}$$

exists. (Note that B implies that μ_t is a continuous measure.)

   C. $\psi(\cdot,s,t) \in C[r_1,r_2]$ for each $t,s \in [a,b]$. $\phi(\cdot,s,t) \in X$ for each $t,s \in [a,b]$.[4]

   D. $c(\cdot,t)$ is nonpositive for each t and $c(\cdot,t)$ is a strongly continuously differentiable function from $[a,b]$ to X.[5]

   E.   For some $s = s_0$, $\psi(\cdot,t,s_0)$ and $\phi(\cdot,t,s_0)$ are strongly continuously differentiable functions from $[a,b]$ to X.

   F_i. $(i = 1,2)$. $\lim_{x \to r_i} \phi(x,s,t)\mu(x,t) = 0$ for each $s,t \in [a,b]$.

   F_3.   For each s,t, $d\sigma_t/d\sigma_s$ is continuous at points of discontinuity of $\mu(\cdot,t)$. (Note that since condition A is assumed, if $\phi(\cdot,s,t) \in X$, condition F_3 implies that $d\sigma_t/d\sigma_s$ is continuous everywhere on $[r_1,r_2]$.)

   G_i.   There exist positive constants m and M independent of s such that for some $x_i \in [0,r_i)$, $m < |\sigma_s(r_i) - \sigma_s(x_i)| < M$.

   H.   E is true and $\phi^*(\cdot,t,s) = \phi(\cdot,t,s)\mu(\cdot,s)$ is a strongly continuously differentiable function from $[a,b]$ to X for some $s = s_0$ with $\phi_t^*(\cdot,t,s_0) = \phi_t(\cdot,t,s_0)\mu(\cdot,s_0)$.

   J.   E is true and $\phi_t(\cdot,t,s_0)\mu(\cdot,s_0) \in X$ for each t.[6]

3. **Conditions under which domain of $D_{\mu(t)}D_{\sigma(t)}$ is independent of t.** In this section and through §5 we shall assume that $c(x,t) \equiv 0$ so that (0.1) is the reduced operator $\Omega(t) = D_{\mu(t)}D_{\sigma(t)}$. Let us now exhibit conditions under which the domain of $\Omega(t) = D_{\mu(t)}D_{\sigma(t)}$ is independent of t for various types of boundary combinations.

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   [4] A function g will be considered to be in X if $g \in C[r_1,r_2]$ and if $\lim_{x \to r_i} g(x)$ exists and has the appropriate value as specified in Definition 2.1.

   [5] The vector-valued function $c(\cdot,t)$ is a strongly differentiable function from $[a,b]$ to X if, and only if, there exists a function $c_t(\cdot,t) \in X$ such that $\lim_{h \to 0} \max_{x \in [r_1,r_2]} |(c(x,t+h) - c(x,t))/h - c_t(x,t)| = \lim_{h \to 0} \|(c(\cdot,t+h) - c(\cdot,t))/h - c_t(\cdot,t)\| = 0$. $c_t(\cdot,t)$ is strongly continuous if, and only if $\lim_{t \to t_0} \max_{x \in [r_1,r_2]} |c_t(x,t) - c_t(x,t_0)| = \lim_{t \to t_0} \|c_t(\cdot,t) - c_t(\cdot,t_0)\| = 0$.

   [6] Note that $H \to J$ and $H \to E$.

LEMMA 3.1. *If both boundaries are regular, both are entrance, or one boundary is regular and one is entrance, conditions* B, C *and* F_3 *imply condition* C_2: *the domain of* $\Omega(t) = D_{\mu(t)}D_{\sigma(t)}$ *is independent of* t.

Proof. It suffices to show that $u \in \text{Dmn}\, \Omega(t)$ implies that $u \in \text{Dmn}\, \Omega(s)$ for $t \neq s$ and $t, s \in [a, b]$. Let

$$(3.1) \qquad\qquad \Omega(t)u = f \in X.$$

We have

$$\Omega(s)u = D_{\mu(s)}D_{\sigma(s)}u$$

$$(3.2) \qquad\qquad = \frac{d\mu_t}{d\mu_s} \cdot D_{\mu(t)}\left(D_{\sigma(s)}u\right)$$

$$= \frac{d\mu_t}{d\mu_s} \cdot D_{\mu(t)}\left(D_{\sigma(t)}u \cdot \frac{d\sigma_t}{d\sigma_s}\right).$$

So

$$(3.3) \qquad \Omega(s)u(x) = \psi(x, s, t) \cdot \Omega(t)u(x) + \phi(x, s, t) \cdot D_{\sigma(t)}u(x)$$

or

$$(3.4) \qquad \Omega(s)u = \psi(\cdot, s, t)f + \phi(\cdot, s, t)D_{\sigma(t)}u.$$

Since $\psi(\cdot, s, t) \in C[r_1, r_2]$ and $f \in X$ by hypothesis, the problem of showing that the $\text{Dmn}\, \Omega(t)$ is independent of t reduces to that of showing that $\phi(\cdot, s, t)D_{\sigma(t)}u$ is an element of X. Let us obtain a convenient expression for $\phi(x, s, t)D_{\sigma(t)}u(x)$ by considering the differential equation (3.1). This equation may be written

$$(3.5) \qquad\qquad D_{\mu(t)}D_{\sigma(t)}u(x) = f(x).$$

We may integerate over $(0, x]$ where $x \in (r_1, r_2)$ and obtain

$$(3.6) \qquad \phi(x, s, t)D_{\sigma(t)}u(x) = \phi(x, s, t)D_{\sigma(t)}u(0) + \phi(x, s, t)\int_0^x f(y)d\mu_t.$$

Since $f \in X$, the function defined for each s, t by the second term on the right of (3.6) will be continuous on the open interval (r_1, r_2) if we can show that

$$\phi(\cdot, s, t)\mu(\cdot, t) \in C(r_1, r_2)$$

for each s, t. With that aim in mind, note that $\phi(\cdot, s, t)\mu(\cdot, t)$ is continuous on (r_1, r_2) except possibly at points of discontinuity of $\mu(\cdot, t)$. These discontinuities form a countable set of jumps and since $d\mu_t/d\mu_s$ exists, for each $t \neq s$ $\mu(\cdot, t)$ and $\mu(\cdot, s)$ are discontinuous at the same points. By definition

$$\phi(x, s, t) = \frac{d\mu_t}{d\sigma_s} \cdot D_{\mu(t)}\left(\frac{d\sigma_t}{d\sigma_s}\right)$$

$$= D_{\mu(s)}\left(\frac{d\sigma_t}{d\sigma_s}\right);$$

but by hypothesis $d\sigma_t/d\sigma_s$ is continuous at points of discontinuity of $\mu(\cdot, t)$ and hence of $\mu(\cdot, s)$ so $\phi(x, s, t) = 0$ at such points. It follows that for each s, t, $\phi(\cdot, s, t)\mu(\cdot, t) \in C(r_1, r_2)$ and the function defined by the second term on the right of (3.6) is in $C(r_1, r_2)$. The function defined for each s, t by the first term on the right of (3.6) is in X by hypothesis. Consequently $\phi(\cdot, s, t)D_{\sigma(t)}u$ is continuous on the open interval. To show that $\phi(\cdot, s, t)D_{\sigma(t)}u$ is actually in X, we have only to show that $\lim_{x \to r_i} \int_0^x f(y)\, d\mu_t$ exists. This is clearly true since $K_i(t) < +\infty$ for r_i regular or entrance; and $K_i(t) < +\infty$ implies $(-1)^i \int_0^{r_i} d\mu_t < +\infty$ which in turn implies that for $f \in C[r_1, r_2]$, $(-1)^i \int_0^{r_i} |f(y)|\, d\mu_t < +\infty$. Thus $\phi(\cdot, s, t)D_{\sigma(t)}u \in X$, and it follows that $\Omega(s)u \in X$.

The argument just used to show that $\phi(\cdot, s, t)D_{\sigma(t)}u$ is in X breaks down if r_i is an exit or natural boundary since $K_i(t) = +\infty$ in these cases. Consequently we shall need to impose an additional condition on $\phi(\cdot, s, t)$ for exit or natural boundaries. This gives rise to the following lemma.

LEMMA 3.2. (a) *If both boundaries are exit, both are natural, or one boundary s natural and one is exit, conditions B, C, F_1, F_2 and F_3 imply condition C_2: the domain of $\Omega(t) = D_{\mu(t)}D_{\sigma(t)}$ is independent of t.*

(b) *If r_i is natural or exit and if r_j ($i \neq j$; $i, j = 1, 2$) is regular or entrance, conditions B, C, F_i, and F_3 imply condition C_2: the domain of $\Omega(t) = D_{\mu(t)}D_{\sigma(t)}$ is independent of t.*

Proof. For $u \in \text{Dmn}\,\Omega(t)$, let $\Omega(t)u = f \in X$. Then as in the proof of Lemma 3.1, we have equations (3.4) and (3.6). Again examination of the right member of (3.6) shows that we have $\phi(\cdot, s, t)D_{\sigma(t)}u$ continuous on the open interval. The function defined for each s, t by the first term on the right of (3.6) is an element of X by hypothesis. Moreover, if F_i is true, for each s, t the function defined by the second term on the right of (3.6) is also an element of X. Thus in all boundary classifications, conditions B, C, F_1, F_2 and F_3 imply that the domain of $\Omega(t) = D_{\mu(t)}D_{\sigma(t)}$ is independent of t.

If r_j happens to be regular or entrance and r_i ($i \neq j$) is exit or natural, however, we need only the conditions B, C, F_i and F_3 to prove the domain is independent of t since we saw in Lemma 3.1 that $\phi(\cdot, s, t)D_{\sigma(t)}u$ is forced to be continuous at a regular or entrance endpoint by conditions B and C.

4. Conditions implying C_3 for regular and/or entrance boundaries with reduced operator $\Omega(t) = D_{\mu(t)}D_{\sigma(t)}$. Now let us consider the operator $C(t, s) = [I - \Omega(t)] \cdot [I - \Omega(s)]^{-1}$ and for various combinations of boundaries let us exhibit conditions implying condition C_3: *$C(t, s)$ is strongly continuously differentiable in t for some s when $\Omega(t)$ is the reduced operator $D_{\mu(t)}D_{\sigma(t)}$.*

First note that if $\text{Dmn}\,\Omega(t)$ is independent of t, then for each t and each s, $C(t, s)$ is an element of $E(X)$, the Banach algebra of endomorphisms on X. The boundedness follows from Lemma 2 of T. Kato [9], and $f \in X$ implies that

$C(t, s)f \in X$ since $F(s, \cdot) = [I - \Omega(s)]^{-1}f$ is an element of $\mathrm{Dmn}\,\Omega(s)$. Also note that for $f \in X$, $C(t,s)f$ can be expressed as follows:

$$(4.1) \qquad C(t,s)f = F(s, \cdot) - \psi(\cdot, t, s)\Omega(s)F(s, \cdot) - \phi(\cdot, t, s)D_{\sigma(s)}F(s, \cdot)$$

$$(4.2) \qquad = F(s, \cdot) - \psi(\cdot, t, s)[F(s, \cdot) - f] - \phi(\cdot, t, s)D_{\sigma(s)}F(s, \cdot).$$

Assuming that the strong derivatives $\psi_t(\cdot, t, s)$ and $\phi_t(\cdot, t, s)$ of $\psi(\cdot, t, s)$ and $\phi(\cdot, t, s)$ exist for some fixed s, define the operator $C_t(t, s)$ so that for the fixed s,

$$(4.3) \qquad C_t(t,s)f = -\psi_t(\cdot, t, s)[F(s, \cdot) - f] - \phi_t(\cdot, t, s)D_{\sigma(s)}F(s, \cdot).$$

The object of the next two lemmas is to show conditions under which this operator $C_t(t, s)$ is actually the strong derivative of $C(t, s)$ with respect to t for the fixed s and is strongly continuous.

LEMMA 4.1. *If $\Omega(t) = D_{\mu(t)}D_{\sigma(t)}$ and both boundaries are regular, both are entrance or one boundary is regular and one is entrance, conditions A, B, C, E, F_3 and J imply condition C_3: $C(t,s) = [I - \Omega(t)][I - \Omega(s)]^{-1}$ is strongly continuously differentiable in t for some s.*

Proof. Fix s at the value for which $\psi_t(\cdot, t, s)$ and $\phi_t(\cdot, t, s)$ exist as assumed in E. First we shall show that $C_t(t, s)$ as defined in (4.3) is actually an endomorphism on X, for each t and some s. Then we shall show that $C_t(t, s)$ is the strong derivative of $C(t, s)$ with respect to t for our fixed s and is strongly continuous.

To show that $C_t(t, s)$ is bounded, consider equation (4.3). Notice that the first term on the right of (4.3) is bounded for each t and the fixed s. Indeed $\| \psi_t(\cdot, t, s) \|$ is bounded in t and for $f \in X$,

$$(4.4) \qquad \| \psi_t(\cdot, t, s)[F(s, \cdot) - f] \| \leqq 2 \| \psi_t(\cdot, t, s) \| \cdot \| f \|$$

inasmuch as $\Omega(s)$ has property S on X. To complete the proof that $C_t(t, s)$ is bounded for each t, we show that for each t and our fixed s there exists a constant $K(s) > 0$ such that

$$(4.5) \qquad \| \phi_t(\cdot, t, s)D_{\sigma(s)}F(s, \cdot) \| \leqq K(s) \| f \|.$$

To this end let us obtain a convenient expression for $D_{\sigma(s)}F(s, \cdot)$ by integrating the equation

$$(4.6) \qquad D_{\mu(s)}D_{\sigma(s)}F(s, x) = F(s, x) - f(x)$$

over $(x, r_2]$ (analogous considerations may be made for r_1). We have

$$(4.7) \qquad D_{\sigma(s)}F(s, x) = \lim_{x \to r_2} D_{\sigma(s)}F(s, x) - \int_x^{r_2} [F(s, y) - f(y)]d\mu_s.$$

In (4.7) $\lim_{x \to r_2} D_{\sigma(s)}F(s, x)$ exists because r_2 is either regular or entrance. There exists a positive constant $N_2(s)$ such that for our fixed s, the second term on the right of (4.7) clearly satisfies the inequality

$$(4.8) \qquad \left| \int_x^{r_2} [F(s, y) - f(y)]d\mu_s \right| \leqq N_2(s) \| f \|$$

inasmuch as $\| F(s, \cdot) \| \leqq \| f \|$ and $(-1)^t \int_0^{r_t} d\mu_s$ exists finite. To show that the first term on the right of (4.7) satisfies a similar inequality, let us obtain an expression for $D_{\sigma(s)}F(s, x)$ by writing out the solution of the nonhomogeneous equation (4.6) in terms of the Green function and then differentiating with respect to σ_s. From H. McKean [11] (analogue of W. Feller [4]) we have

$$(4.9) \qquad F(s, x) = \int_{r_1}^{r_2} G(x, y; s)f(y)d\mu_s$$

where

$$(4.10) \qquad G(x, y; s) = \frac{u_1(x, s)u_2(y, s)}{\omega(u_2(y, s), u_1(y, s))} \qquad \text{for } y \geqq x$$

$$= \frac{u_1(y, s)u_2(x, s)}{\omega(u_2(y, s), u_1(y, s))} \qquad \text{for } y < x.$$

Here (i) u_1 is positive strictly increasing and u_2 is positive strictly decreasing, (ii) u_1 and u_2 are solutions of the homogeneous equation $[I - \Omega(s)]u = 0$, (iii) u_i satisfies regular or entrance boundary conditions depending on whether r_i is regular or entrance, and (iv) $\lim_{x \to r_i} D_{\sigma(s)}u_i(x, s) < +\infty$. Moreover from W. Feller [3] we learn that the wronskian $\omega(u_2(y, s), u_1(y, s)) = D_{\sigma(s)}u_1(x)u_2 - u_1 D_{\sigma(s)}u_2(x)$ is a positive constant $k(s)$. Without loss of generality we can consider the wronskian $\omega(u_2, u_1)$ to be normed so that $k(s) = 1$. Therefore,

$$(4.11) \qquad F(s, x) = u_2(x, s)\int_{r_1}^x u_1(y, s)f(y)d\mu_s + u_1(x, s)\int_x^{r_2} u_2(y, s)f(y)d\mu_s.$$

Now we can compute $D_{\sigma(s)}F(s, x)$ from (4.11) as follows:

$$(4.12) \quad D_{\sigma(s)}F(s, x) = D_{\sigma(s)}u_2(x, s)\int_{r_1}^x u_1(y, s)f(y)d\mu_s + D_{\sigma(s)}u_1(x, s)\int_x^{r_2} u_2(y, s)f(y)d\mu_s.$$

An analysis of all the possible choices of the constant p_2 in the definition of the boundary conditions ([7]) shows that if r_2 is regular and r_1 is entrance or regular, either

$$(4.13) \qquad \lim_{x \to r_2} D_{\sigma(s)}F(s, x) = 0$$

or

$$(4.14) \qquad \lim_{x \to r_2} D_{\sigma(s)}F(s, x) = -n\int_{r_1}^{r_2} u_1(y, s)f(y)d\mu_s$$

where $n > 0$, and $u_1(x, s)$ is a positive strictly increasing solution of $y - \Omega(s)y = 0$ such that $\lim_{x \to r_2} u_1(x, s)$ and $\lim_{x \to r_2} D_{\sigma(s)}u_1(x, s)$ are both finite. In either case if r_2 is regular, there exists a positive constant $N_1(s)$ such that

---

([7]) Complete details are given in [12].

$$(4.15) \qquad \left| \lim_{x \to r_2} D_{\sigma(s)} F(s, x) \right| \leqq N_1(s) \|f\|$$

for our fixed s.

It follows from (4.7), (4.8), and (4.15) that if r_2 is regular and r_1 is entrance or regular, there exists $N(s) > 0$ such that for our fixed s,

$$(4.16) \qquad \left\| D_{\sigma(s)} F(s, \cdot) \right\| \leqq N(s) \|f\|.$$

Consequently the boundedness condition (4.5) holds and so for r_2 regular and r_1 entrance or regular $C_t(t, s)$ *is* bounded for each t and our fixed s. To complete the proof of the boundedness of $C_t(t, s)$ it remains to consider the case in which r_2 is an entrance boundary. In this case, since $F(s, \cdot) \in \mathrm{Dmn} \, \Omega(s)$ and hence satisfies appropriate boundary conditions, $\lim_{x \to r_2} D_{\sigma(s)} F(s, x) = 0$ and again the boundedness condition (4.5) holds and forces $C_t(t, s)$ to be bounded for each t and our fixed s.

To see that $C_t(t, s)$ is an endomorphism on X, it remains to show that for $f \in X$, $C_t(t, s) f \in X$. We see from (4.3) that, to show this, it suffices to note that $\phi_t(\cdot, t, s) D_{\sigma(s)} F(s, \cdot) \in X$.

To demonstrate that $C(t, s)$ is strongly continuously differentiable in t for some s, we must show that $C_t(t, s)$ actually *is* the strong derivative of $C(t, s)$ with respect to t for our fixed s and we must show that $C_t(t, s)$ is strongly continuous in t for our fixed s. In other words we must show that

$$(4.17) \qquad \lim_{h \to 0} \left\| \left[\frac{C(t+h, s) - C(t, s)}{h} - C_t(t, s) \right] f \right\| = 0$$

and that

$$(4.18) \qquad \lim_{t \to t_0} \left\| \left[C_t(t, s) - C_t(t_0, s) \right] f \right\| = 0$$

both for all $f \in X$ and the fixed s. Computing from (4.2) and (4.3) we have for any $f \in X$,

$$0 \leqq \left\| \left[\frac{C(t+h, s) - C(t, s)}{h} - C_t(t, s) \right] f \right\|$$

$$(4.19) \qquad \leqq \left\| \frac{\psi(\cdot, t+h, s) - \psi(\cdot, t, s)}{h} - \psi_t(\cdot, t, s) \right\| \, \|F(s, \cdot) - f\|$$

$$+ \left\| \frac{\phi(\cdot, t+h, s) - \phi(\cdot, t, s)}{h} - \phi_t(\cdot, t, s) \right\| \, \left\| D_{\sigma(s)} F(s, \cdot) \right\|$$

and

$$0 \leqq \left\| \left[C_t(t, s) - C_t(t_0, s) \right] f \right\|$$

$$(4.20) \qquad \leqq \left\| \psi_t(\cdot, t, s) - \psi_t(\cdot, t_0, s) \right\| \, \|F(s, \cdot) - f\|$$

$$+ \left\| \phi_t(\cdot, t, s) - \phi_t(\cdot, t_0, s) \right\| \, \left\| D_{\sigma(s)} F(s, \cdot) \right\|.$$

Since $\| F(s, \cdot) - f \| \leqq 2 \| f \|$ and (4.16) holds, it follows from (4.19), (4.20) and condition E that $C(t, s)$ is strongly continuously differentiable in t for some s. In fact, it follows that $C(t, s)$ is even uniformly differentiable and that $C_t(t, s)$ is even uniformly continuous.

5. Conditions implying C_3 for one exit or one regular boundary with reduced operator $\Omega(t) = D_{\mu(t)} D_{\sigma(t)}$.

LEMMA 5.1. (a) *If $\Omega(t) = D_{\mu(t)} D_{\sigma(t)}$ and r_i is an exit boundary and r_j $(i \neq j;$ $i, j = 1, 2)$ is exit or natural, then conditions* A, C, F_1, F_2, F_3, G_i *and* H *imply condition* C_3: $C(t, s) = [I - \Omega(t)][I - \Omega(s)]^{-1}$ *is strongly continuously differentiable in t for some s.*

(b) *If $\Omega(t) = D_{\mu(t)} D_{\sigma(t)}$ and r_i is an exit boundary and r_j $(i \neq j; i, j = 1, 2)$ is regular or entrance, then conditions* A, C, F_i, F_3, G_i *and* H *imply condition* C_3.

In both case (a) and case (b) the domain of $\Omega(t)$ is independent of t (Lemma (3.2)) so as in §4, $C(t, s)$ is in an endomorphism on X for each t, s. Fix s at that value for which E and H hold. As before for $f \in X$, we have equations (4.2) and (4.3). The proof of Lemma (5.1) proceeds in much the same manner as the proof of Lemma (4.1). Again we nominate a candidate $C_t(t, s)$ for the strong continuous derivative of $C(t, s)$. Again, we show that $C_t(t, s)$ is an element of $E(X)$, is the strong derivative of $C(t, s)$, and is such that $C_t(t, s)$ is strongly continuous.

In this case, however, we work with a different expression for $C_t(t, s)f$ obtained by replacing $D_{\sigma(s)} F(s, \cdot)$ in (4.3) by an equivalent taken from equation (4.17). Explicitly, from (4.17) we have,

$$(5.1) \qquad D_{\sigma(s)} F(s, x) = D_{\sigma(s)} F(s, 0) + \int_0^x [F(s, y) - f(y)] d\mu_s.$$

So (4.3) yields the following expression for $C_t(t, s)f(x)$:

$$
\begin{aligned}
C_t(t, s)f(x) = \ & -\psi_t(x, t, s)[F(s, x) - f(x)] \\
(5.2) \qquad & -\phi_t(x, t, s) D_{\sigma(s)} F(s, 0) \\
& -\phi_t(x, t, s) \int_0^x [F(s, y) - f(y)] d\mu_s.
\end{aligned}
$$

Again the proof of the boundedness of $C_t(t, s)$ for the fixed s poses the biggest problem. In demonstrating this boundedness we integrate equation (5.1) over $(x, r_2]$ where $x \in (0, r_2)$ is equal to the x_2 of condition G_2 (analogous considerations may be made for r_1) and obtain

$$
\begin{aligned}
(5.3) \qquad \lim_{y \to r_2} \ F(s, y) - F(s, x) = \ & \int_x^{r_2} D_{\sigma(s)} F(s, 0) d\sigma_s \\
& + \int_x^{r_2} \int_0^x [F(s, y) - f(y)] d\mu_s d\sigma_s.
\end{aligned}
$$

Since r_2 is exit, the integrals on the right of (5.3) exist and $\lim_{y \to r_2} F(s, y) = 0$. So (5.3) yields

(5.4) $\qquad D_{\sigma(s)} F(s, 0) = \dfrac{- F(s, x) - \int_x^{r_2} \int_0^x [F(s, y) - f(y)] d\mu_s d\sigma_s}{\int_x^{r_2} d\sigma_s}.$

This expression for $D_{\sigma(s)} F(s, 0)$ can be used in (5.2), and the hypotheses of the lemma are readily seen to force the boundedness of $C_t(t, s)$ for the fixed s. If r_1 is the exit boundary instead of r_2, we integrate equation (5.1) over $[r_1, x)$ where $x \in (r_1, 0)$ is the x_1 of condition G_1.

The fact that $C_t(t, s)$ *is the strong derivative of* $C(t, s)$ *and is strongly continuous* follows directly from (5.2) since ψ and ϕ are strongly continuously differentiable.

As in Lemma 4.1, the method of proof used shows that $C_t(t, s)$ is the uniform derivative of $C(t, s)$ and that $C_t(t, s)$ is uniformly continuous.

COROLLARY. (a) *If* $\Omega(t) = D_{\mu(t)} D_{\sigma(t)}$ *and* r_i *is regular, and if* r_j $(i \neq j; i, = 1, 2)$ *is exit or natural, then conditions* A, C, F_j, F_3, G_i *and* H *imply* C_3.

(b) *If* $\Omega(t) = D_{\mu(t)} D_{\sigma(t)}$ *and* r_i *is regular, and if* r_j $(i \neq j; i, j = 1, 2)$ *is regular or entrance, then conditions* A, C, F_3, G_i, *and* H *imply* C_3.

Here equation (5.4) is replaced by

(5.4)′ $\qquad D_{\sigma(s)} F(s, 0) = \dfrac{F(s, r_2) - F(s, x) - \int_x^{r_2} \int_0^x [F(s, y) - f(y)] d\mu_s d\sigma_s}{\int_x^{r_2} d\sigma_s}$

and the proof is analogous to that of Lemma 5.1.

6. The general operator. Let us now direct our attention to the general operator

(6.1) $\qquad\qquad\qquad \Omega(t) = D_{\mu(t)} D_{\sigma(t)} + c(x, t)$

where $c(x, t)$ satisfies *condition* D: *for each* t, $c(\cdot, t) \leq 0$ *and is continuous on the closed interval* $[r_1, r_2]$, *and* $c(\cdot, t)$ *is strongly continuously differentiable in* t *with strong derivative* $c_t(\cdot, t) \in X$ *for each* t. By slightly modifying the proofs of Lemmas 3.1, 3.2, 4.1 and 5.1, one can readily prove that these lemmas are also true for the general operator (6.1). Indeed if $c(\cdot, t) \leq 0$, the key equations used in the proofs of these lemmas are modified by the addition of one to three terms involving $c(\cdot, t)$, and these additional terms fall in line with the proofs presented by virtue of the fact that $c(x, t)$ satisfies condition D. For example, if $c(\cdot, t) \neq 0$, equation (3.3) is replaced by

(3.3)′ $\qquad \Omega(s)u = \psi(\cdot, s, t) D_{\mu(t)} D_{\sigma(t)} u + \phi(\cdot, s, t) D_{\sigma(t)} u + c(\cdot, s)u,$

and the key expressions for $C(t, s)f$ and $C_t(t, s)f$ become

(4.2)′ $\qquad C(t, s)f = F(s, \cdot) - \psi(\cdot, t, s)[F(s, \cdot) - f] - \phi(\cdot, t, s) D_{\sigma(s)} F(s, \cdot)$
$\qquad\qquad\qquad + \psi(\cdot, t, s) c(\cdot, s) F(s, \cdot) - c(\cdot, t) F(s, \cdot)$

and

$$(4.3)' \quad C_t(t,s)f = -\psi_t(\cdot,t,s)[F(s,\cdot) - f] - \phi_t(\cdot,t,s)D_{\sigma(s)}F(s,\cdot)$$
$$+ \psi_t(\cdot,t,s)c(\cdot,s)F(s,\cdot) - c_t(\cdot,t)F(s,\cdot).$$

In summary we see that the following theorem is true.

THEOREM (6.1). *If r_1 and r_2 are each regular, entrance, or exit boundaries, or if one boundary is exit and one natural, or one is regular and one natural, then in each case any appropriate set of conditions given in Lemmas (4.1) and (5.1) which implies that $C(t,s) = [I - \Omega(t)][I - \Omega(s)]^{-1}$ is strongly continuously differentiable in t for some s also implies that there exists a solution of the equation*

$$u_t(x,t) = D_{\mu(t)}D_{\sigma(t)}u(x,t) + c(x,t)u(x,t)$$

where for each $t \in [a,b]$, $u(\cdot,t)$ is an element of the Banach space X of Definition 2.1 and where $c(x,t)$ satisfies condition D. The solution is uniquely determined by the initial value $u(x,a)$ in X.

7. **The special case of** $A(t) = a(x,t)d^2/dx^2 + b(x,t)d/dx$. Let us now consider the special case of the operator $A(t) = a(x,t)d^2/dx^2 + b(x,t)d/dx$ where for each t, $a(x,t) > 0$ and $b*(x,t) = b(x,t)/a(x,t)$ is continuous on the open interval (r_1,r_2) with $0 \in (r_1,r_2)$. In particular let us see what the boundary classifications and boundary conditions look like in this special case, and let us discover what restrictions the conditions of Theorem (6.1) impose on the coefficients of the equation $u_t(x,t) = A(t)u(x,t)$.

As in §2 let

$$(7.1) \qquad B(x,t) = \int_0^x b*(s,t)ds,$$

$$(7.2) \qquad \sigma(x,t) = \int_0^x e^{-B(s,t)}ds$$

and

$$(7.3) \qquad \mu(x,t) = \int_0^x \frac{1}{a(s,t)}e^{B(s,t)}ds.$$

Both $\sigma(x,t)$ and $\mu(x,t)$ are obviously monotone strictly increasing and $\sigma(x,t)$ is continuous on (r_1,r_2) for each t since $b*(\cdot,t) \in C(r_1,r_2)$ for each t. The $J_i(t)$ and the $K_i(t)$ used in the boundary classifications become

$$(7.4) \qquad J_i(t) = (-1)^i \int_0^{r_i} e^{-B(x,t)} \int_0^x \frac{1}{a(s,t)}e^{B(s,t)}ds\,dx$$

and

$$(7.5) \qquad K_i(t) = (-1)^i \int_0^{r_i} \frac{1}{a(x,t)}e^{B(x,t)} \int_0^x e^{-B(s,t)}ds\,dx.$$

In this case for each t

$$K_i(t) < + \infty \to (-1)^i \int_0^{r_i} \frac{1}{a(x,t)} e^{B(x,t)} dx < + \infty$$

and

$$J_i(t) < + \infty \to (-1)^i \int_0^{r_i} e^{-B(x,t)} dx < + \infty.$$

The classical boundary conditions become

$$\lim_{x \to r_i} \{(1 - p_i)u(x,t) + (-1)^i p_i u_x(x,t) e^{B(x,t)}\} = 0 \text{ for } r_i \text{ regular,}$$

(7.6)
$$\lim_{x \to r_i} u(x,t) = 0 \text{ for } r_i \text{ exit,}$$

$$\lim_{x \to r_i} u_x(x,t) e^{B(x,t)} = 0 \text{ for } r_i \text{ entrance,}$$

$$\lim_{x \to r_i} u(x,t) = 0 \text{ for } r_i \text{ natural}$$

where $i = 1$ or 2 and $p_i \in [0,1]$.

From equations (7.2) and (7.3) we see that conditions B, A, and F_3 stated in §2 are satisfied and in fact

(7.7)
$$\frac{d\mu_s}{d\sigma_s} = \frac{1}{a(x,s)} e^{2B(x,s)},$$

(7.8)
$$\frac{d\mu_t}{d\mu_s} = \frac{a(x,s)}{a(x,t)} e^{B(x,t)} e^{-B(x,s)},$$

and

(7.9)
$$\frac{d\sigma_t}{d\sigma_s} = e^{B(x,s)} e^{-B(x,t)}.$$

Consequently

(7.10)
$$\phi(\cdot,s,t) = a(\cdot,s) e^{-B(\cdot,t)} [b^*(\cdot,s) - b^*(\cdot,t)],$$

(7.11)
$$\psi(\cdot,s,t) = \frac{a(\cdot,s)}{a(\cdot,t)},$$

(7.12)
$$\phi(\cdot,s,t) \int_0^x d\mu_t = \phi(\cdot,s,t) \int_0^x \frac{1}{a(v,t)} e^{B(v,t)} dv,$$

and

(7.13)
$$\int_{x_0^2}^r d\sigma_s = \int_{x_0^2}^r e^{-B(x,s)} dx.$$

Define the condition

A_0. *For each t, $a(\cdot,t)$ is positive and $b^*(\cdot,t)$ is continuous on (r_1,r_2).*
Then referring to the conditions C_1, C_2, C_3 listed in §1 and the conditions on μ_t and σ_t listed in §2, the main results established for $\Omega(t) = D_{\mu(t)} D_{\sigma(t)}$ may be specialized for $A(t) = a(x,t) d^2/dx^2 + b(x,t) d/dx$ as follows.

LEMMA 7.1. *In all boundary classifications condition A_0 implies condition C_1: $A(t)$ is defined for $t \in [a,b]$ and has property S (Definition (1.1)) for each t.*

LEMMA 7.2. *If both boundaries are regular, both are entrance, or one boundary is regular and one is entrance, A_0 and C imply C_2: the domain of $A(t)$ is independent of t; and A_0, C and E imply condition C_3: $C(t,s) = [I - A(t)]$ $\cdot [I - A(s)]^{-1}$ is strongly continuously differentiable in t for some s.*

LEMMA 7.3. (a) *If both boundaries are exit, both are natural, or one boundary is natural and one is exit, A_0, C, F_1 and F_2 imply C_2. If r_i is exit and r_j ($i \neq j$; $i,j = 1,2$) is exit or natural, A_0, C, F_1, F_2, G_i and H imply C_3.*

(b) *If r_i is natural or exit and if r_j ($i \neq j$; $i,j = 1,2$) is regular or entrance, A_0, C and F_i imply C_2. If r_i is exit and r_j is regular or entrance, A_0, C, F_i, G and H imply C_3.*

LEMMA 7.4. (a) *If r_i is regular and if r_j ($i \neq j$) is exit or natural, A_0, C, F_j, G_i and H imply C_3.*

(b) *If r_i is regular and if r_j is regular or entrance, A_0, C, E and G_i imply C_3. In each case each appropriate set of conditions implying C_3 implies the existence of a solution of the equation*

$$u_t(x, t) = a(x, t)u_{xx}(x, t) + b(x, t)u_x(x, t)$$

uniquely determined by the initial value $u(x, a)$. Of course if $c(x, t)$ satisfies condition D, then each set of conditions implying C_3 is also a set of conditions implying the existence of a solution of the equation

$$u_t(x, t) = a(x, t)u_{xx}(x, t) + b(x, t)u_x(x, t) + c(x, t)u(x, t)$$

uniquely determined by $u(x, a)$.

8. **Examples.** In each specific example it is necessary to determine whether or not the strong derivatives $\phi_t(\cdot, t, s)$ and $\psi_t(\cdot, t, s)$ exist. This computation is frequently made much easier by using a simple criterion. This criterion asserts that if $\partial^2 g/\partial t^2$ is continuous on $[r_1, r_2]$ and bounded uniformly in x and t, then the derivative $\partial g/\partial t$ exists as a strong derivative.

EXAMPLE 1. As an example of a particular equation $u_t = A(t)u$ with coefficients which satisfy the conditions of Theorem (6.1) even though $b(x, t)$ is discontinuous at both boundaries, consider the equation

$$(8.1) \qquad u_t(x, t) = u_{xx}(x,t) + \left(\frac{-x}{1 - x^2} + 3t(1 - x^2) \right) u_x(x, t)$$

for $x \in [-1, 1]$. Here

$$e^{B(x,t)} = (1 - x^2)^{1/2} e^{t(3x - x^3)}.$$

We see that both $e^{B(x,t)}$ and $e^{-B(x,t)}$ are in $L(0, 1)$ and in $L(-1, 0)$ so that 1 and -1 are both regular boundaries. Choose the boundary conditions so that X is $C[-1, 1]$.

$b(x, t)$ is clearly continuous on the open interval $(-1, 1)$ for each t so condition A (§2) is satisfied. We have

$$\phi(x, s, t) = 3(1 - x^2)^{1/2} e^{-t(3x - x^3)} (s - t) \text{ and } \psi(x, t, s) \equiv \psi(x, s, t) \equiv 1.$$

$\phi(\cdot, s, t)$ and $\psi(\cdot, s, t)$ are both continuous on the closed interval $[-1, 1]$ so condition C (§2) is satisfied. Both $\phi_t(\cdot, t, s)$ and $\psi_t(\cdot, t, s)$ are in $C[-1, 1]$ for the fixed s and condition E (§2) is satisfied.

EXAMPLE 2. An example of the case in which $b(x, t)$ is discontinuous at an entrance boundary is given by the equation

$$(8.2) \qquad u_t(x, t) = u_{xx}(x, t) + \left(\frac{1}{x - 1} + 2h(t)(x - 1) \right) u_x(x, t)$$

where $x \in [-1, 1]$, $h(t) \geq 0$ for each $t \in [a, b]$, and $d^2 h/dt^2$ is continuous. Here

$$e^{B(x, t)} = (x - 1) e^{h(t)(x^2 - 2x)}.$$

Clearly $e^{B(x, t)}$ is an element of both $L(-1, 0)$ and $L(0, 1)$. Also $e^{-B(x, t)} \in L(-1, 0)$. However, $e^{-B(x, t)}$ is not an element of $L(0, 1)$. Thus, -1 is a regular boundary and $+1$ is either entrance or natural. In fact, it is readily seen that $e^{B(x, t)} \int_0^x e^{-B(y, t)} dy$ is in $L(0, 1)$ so $+1$ is an entrance boundary.

Condition A_0 (§7) is certainly true, and

$$\phi(x, s, t) = 2e^{-h(t)(x^2 - 2x)}[h(s) - h(t)],$$

$$\psi(x, s, t) \equiv \psi(x, t, s) \equiv 1.$$

So, if we choose the boundary conditions so that X is $C[-1, 1]$, $\phi(\cdot, s, t)$ and $\psi(\cdot, s, t)$ are both in X for each s, t. Also for fixed s, by the strong differentiability criterion

$$\phi_t(x, t, s) = 2e^{-h(s)(x^2 - 2x)} h_t(t)$$

and

$$\psi_t(\cdot, t, s) = 0$$

so $\phi_t(\cdot, t, s)$ and $\psi_t(\cdot, t, s)$ are both in X for each t and the fixed s. It follows that there exists a solution of (8.2) uniquely determined by the initial value.

EXAMPLE 3. The case in which $b(x, t)$ is discontinuous at an exit boundary may be illustrated by the equation

$$(8.3) \qquad u_t(x, t) = u_{xx}(x, t) + \left(\frac{1}{1 - x} + 2h(t)(1 - x) \right) u_x(x, t)$$

where $x \in [-1, 1]$, $t \in [a, b]$, $a \geq 0$, and $h(t) \geq 0$ for each t with $d^2 h/dt^2$ continuous. Here both $e^{B(x, t)}$ and $e^{-B(x, t)}$ are elements of $L(-1, 0)$ so -1 is regular. $e^{-B(x, t)}$ is also an element of $L(0, 1)$, but $e^{B(x, t)} \notin L(0, 1)$. Thus $+1$ is either exit or natural. Since it may be shown that $\int_0^1 e^{-B(x, t)} \int_0^x e^{B(y, t)} dy dx < \infty$, $+1$ is actually an exit boundary.

The conditions which must be true in order for Theorem (6.1) to hold for equation (8.3) are A_0 (§7), and C, E, F_2, G_1 and H (§2). It may be readily shown that these conditions are all true in this example.

EXAMPLE 4. Now let us examine an example of an operator $A(t)$ whose coefficient $b(x, t)$ has a jump discontinuity at some point within $[r_1, r_2]$ but which is such that the existence and uniqueness of the solution of $u_t = A(t)u$ with initial value $u(x, a)$ can be ascertained without piecing together two solutions. Let $x \in [-1, 1]$ and $t \in [a, b]$, $a \geqq 0$. Let

$$
\begin{aligned}
\text{(8.4)} \qquad u_t(x, t) &= u_{xx}(x, t) + 2xtu_x(x, t) && \text{for } x < 0 \\
&= u_{xx}(x, t) + (2xt + 3)u_x(x, t) && \text{for } x \geqq 0.
\end{aligned}
$$

Then

$$
\begin{aligned}
B(x, t) &= x^2 t && \text{for } x < 0 \\
&= x^2 t + 3x && \text{for } x \geqq 0.
\end{aligned}
$$

We see that $e^{B(x,t)}$ and $e^{-B(x,t)}$ are both in $L(0, 1)$ and $L(-1, 0)$ so -1 and 1 are both regular boundaries. Let

$$
\begin{aligned}
\sigma_t(x) &= \int_0^x e^{-y^2 t} dy && \text{for } x < 0, \\[2mm]
&= \int_0^x e^{-y^2 t - 3y} dy && \text{for } x \geqq 0, \\[2mm]
\mu(x, t) &= \int_0^x e^{y^2 t} dy && \text{for } x < 0, \\[2mm]
&= \int_0^x e^{y^2 t + 3y} dy && \text{for } x \geqq 0.
\end{aligned}
$$

Then (8.4) is reduced to the canonical form

$$
u_t = D_{\mu(t)} D_{\sigma(t)} u
$$

which holds for *all* $x \in [-1, 1]$.

Note that conditions A and B (§2) are both true. Further

$$
\begin{aligned}
\phi(x, s, t) &= 2x e^{-x^2 t}(s - t) && \text{for } x < 0 \\
&= 2x e^{-x^2 t - 3x}(s - t) && \text{for } x \geqq 0 \\
\psi(x, s, t) &\equiv 1 && \text{for } x \in [-1, 1] \text{ and } s, t \in [a, b].
\end{aligned}
$$

Note that $\lim_{x \to 0^+} \phi(x, s, t) = \lim_{x \to 0^-} \phi(x, s, t) = \phi(0, s, t) = 0$ so $\phi(\cdot, s, t)$ and $\psi(\cdot, s, t)$ are both continuous for all $x \in [-1, 1]$ for each s, t. Thus, if we choose the regular boundary conditions so that X is $C[-1, 1]$, condition C (§2) is true.

For fixed s we have

$$\phi_t(x, t, s) = 2x e^{-x^2 s} \qquad \text{for } x < 0$$
$$= 2x e^{-x^2 s - 3x} \qquad \text{for } x \geq 0$$

and

$$\psi_t(x, t, s) \equiv 0.$$

We see that for the fixed s, $\phi_t(\cdot, t, s)$ and $\psi_t(\cdot, t, s)$ are both continuous for all $x \in [-1, 1]$ for each t so condition E (§2) is satisfied. It follows that there exists a solution to (8.4) uniquely determined by $u(x, a)$.

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